

NON-CONTRACTIBLE PERIODIC TRAJECTORIES OF SYMPLECTIC VECTOR FIELDS, FLOER COHOMOLOGY AND SYMPLECTIC TORSION.

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(preliminary version)

ABSTRACT. For a closed symplectic manifold (M, ω) , a compatible almost complex structure J , a 1-periodic time dependent symplectic vector field Z and a homotopy class of closed curves γ we define a Floer complex based on 1-periodic trajectories of Z in the homotopy class γ . We suppose that the closed 1-form $i_{Z_t}\omega$ represents a cohomology class $\beta(Z) := \beta$, independent of t . We show how to associate to $(M, \omega, \gamma, \beta)$ and to two pairs (Z_i, J_i) , $i = 1, 2$ with $\beta(Z_i) = \beta$ an invariant, the relative symplectic torsion, which is an element in the Whitehead group $\text{Wh}(\Lambda_0)$, of a Novikov ring Λ_0 associated with (M, ω, Z, γ) . If the cohomology of the Floer complex vanishes or if γ is trivial we derive an invariant, the symplectic torsion for any pair (Z, J) .

We prove, that when $\beta(\gamma) \neq 0$, or when γ is non-trivial and β is ‘small’, the cohomology of the Floer complex is trivial, but the symplectic torsion can be non-trivial. Using the first fact we conclude results about non-contractible 1-periodic trajectories of 1-periodic symplectic vector fields. In this version we will only prove the statements for closed weakly monotone manifolds, but note that they remain true as formulated for arbitrary closed symplectic manifolds.

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1. INTRODUCTION

Let (M, ω) be a closed connected symplectic manifold, and $Z \in C^\infty(S^1, \mathfrak{X}(M, \omega))$ a 1-periodic symplectic vector field on M . We are interested in 1-periodic trajectories of Z . If every Z_t is Hamiltonian, Floer theory (initially developed by Floer, Hofer-Salamon for weakly monotone symplectic manifolds and later extended by Liu-Tian, Fukaya-Ono to all symplectic manifolds) was used by Salamon-Zehnder [SZ92] to count the 1-periodic trajectories that are contractible. In fact, when all these trajectories are non-degenerate they generate a cochain complex which computes $H^*(M; \mathbb{Z})$. As a consequence the number of non-degenerate 1-periodic contractible trajectories can be estimated from below by the sum of the Betti numbers of M . In [LO95] this was extended to arbitrary 1-periodic time dependent symplectic vector field Z as above.

The discussion in the above mentioned papers is done under the additional hypotheses, that the symplectic manifold (M, ω) is weakly monotone, see Definition 4.3, but in view of the work of Liu-Tian [LT98] and Fukaya-Ono [FO99] this hypothesis can be removed.

We will go one step further and consider 1-periodic trajectories which are non-contractible under the hypothesis that $\beta := [i_{Z_t}\omega] \in H^1(M; \mathbb{R})$ does not depend on t . We consider:

- (1) a homotopy class γ of closed curves,
- (2) a pair (Z, J) , where Z is a 1-periodic symplectic vector field whose 1-periodic trajectories in γ are all non-degenerate and such that $\beta := [i_{Z_t}\omega] \in H^1(M; \mathbb{R})$ is independent of t , J is a compatible almost complex structure (cf. section 4) and
- (3) additional data, consisting of a c-structure above γ (cf. Definition 2.1) and ‘coherent orientations’.

In this paper we suppose for simplicity that (M, ω) is weakly monotone and the pair (Z, J) satisfies a regularity condition with respect to γ which is a generic property (cf. section 4). In view of the work [LT98] and [FO99] these two hypotheses are not necessary if the Novikov ring associate to $(M, \omega, \beta, \gamma)$ has coefficients containing the rational numbers¹. Under the hypotheses of weak monotonicity and γ -regularity the concept of ‘coherent orientations’ is the one considered in [FH93]. Without these hypotheses ‘coherent orientations’ should be understood as ‘compatible Kuranishi structures with corners’ as discussed in [FO99] section 19.

In sections 3 and 4 we associate to this data a Floer complex, which is a cochain complex of \mathbb{Z}_{2N} -graded free Λ -modules. Here Λ is a Novikov ring associated to $(M, \omega, \beta, \gamma)$, cf. section 2, and the integer N is the minimal Chern number as defined in section 2, depending on (M, ω, γ) .

The underlying free module of this complex is generated by the 1-periodic closed trajectories of Z in the class γ , which are non-degenerate in view of the hypotheses on Z , hence are finitely many.

As expected one shows that the cohomology of this complex is independent of the pair (Z, J) . If $\beta(\gamma) \neq 0$, or if γ is non-trivial and β is ‘small’, this cohomology vanishes. These properties are collected in Theorems 5.7 and 5.8 of which one derives the following geometric result. It remains true as stated without the weak monotonicity hypothesis.

¹in view of more recent work [FO00] not necessary at all.

Theorem 1.1. *Let (M, ω) be a closed connected weakly monotone symplectic manifold and Z a 1-periodic time dependent symplectic vector field, such that the cohomology class $\beta = [i_{Z_t}\omega] \in H^1(M; \mathbb{R})$ is independent of t . Then:*

- (1) *If in a given non-trivial homotopy class of closed curves γ , so that $\beta(\gamma) \neq 0$, there exists a non-degenerate 1-periodic trajectory of Z , then there exists at least one more geometrically different, possibly degenerate trajectory in the same homotopy class.*
- (2) *If all 1-periodic trajectories of Z in a given non-trivial homotopy class γ , so that $\beta(\gamma) \neq 0$, are non-degenerate, then their number is even.*

Remark 1.2. This theorem remains true without the hypothesis $\beta(\gamma) \neq 0$. K. Ono has informed us, that he can also prove this result.

Our next observation is, that the cochain complex associated to $(M, \omega, \gamma, Z, J)$ and the additional data has a well defined torsion, independent of the additional data, at least in the case that $\beta(\gamma) \neq 0$ or γ trivial. This torsion, a priori defined in $\text{Wh}(\Lambda)$, actually lies in the subgroup $\text{Wh}(\Lambda_0)$. The rings Λ and Λ_0 are defined in section 2 and the groups $\text{Wh}(\Lambda)$ and $\text{Wh}(\Lambda_0)$ in section 6.

We hope that this symplectic torsion will be an important new invariant to be used in the study of 1-periodic closed trajectories of Z and maybe other Floer complex related problems, particularly when the Floer complex is acyclic. At present we know little about it (despite of a good number of conjectures we have). For example we know that it is non-trivial, cf. section 8, but we do not know yet how it depends on the almost complex structure J . We also know how to calculate this torsion in the case γ is trivial and Z is time independent. More precisely, consider the Riemannian metric g induced from ω and J cf. section 4. The symplectic torsion can be expressed both in terms of the dynamics of the vector field $\text{grad}_g(i_Z\omega)$, precisely its closed trajectories, and in terms of the spectral theory of the (time dependent) Laplacians of the complex $(\Omega^*(M), d^* + t(i_Z\omega) \wedge \cdot)$ with respect to the metric g . The second expression requires some Dirichlet series theory as in [BH01]. When the cohomology of M with coefficients in $\beta = [i_Z\omega]$ is trivial our symplectic torsion identifies to the ‘zeta function’, defined in [Hu00].

We point out that section 1.9 of [Hu00], entitled ‘Possible generalizations’ refers to ideas close to those presented here. We thank A. Fel’shtyn for bringing this paper to our attention. We will return to this symplectic torsion in future research. We thank K. Ono for pointing out a mistake in a previous version of this paper. We understand, that he was also aware of the possibility to define a Floer complex for non-contractible trajectories.

2. TOPOLOGICAL DATA AND ASSOCIATED NOVIKOV RINGS

Let (M, ω) be a closed connected symplectic manifold of dimension $2n$ and let $\mathcal{L} := C^\infty(S^1, M)$ denote the space of smooth free loops, where we think of S^1 as \mathbb{R}/\mathbb{Z} . The first Chern class $c_1 \in H^2(M; \mathbb{Z})$ of (M, ω) defines a cohomology class $\bar{c}_1 \in H^1(\mathcal{L}; \mathbb{Z}) = \text{Hom}(H_1(\mathcal{L}; \mathbb{Z}), \mathbb{Z})$, by interpreting a class in $H_1(\mathcal{L}; \mathbb{Z})$ as linear combination of ‘tori’, i.e. maps $S^1 \times S^1 \rightarrow M$, and integrating c_1 over it. Similarly $[\omega] \in H^2(M; \mathbb{R})$ gives rise to a cohomology class $\overline{[\omega]} \in H^1(\mathcal{L}; \mathbb{R}) = \text{Hom}(H_1(\mathcal{L}; \mathbb{Z}), \mathbb{R})$. Given a cohomology class $\beta \in H^1(M; \mathbb{R})$ we define

$$\phi := \overline{[\omega]} + \text{ev}^*\beta \in H^1(\mathcal{L}; \mathbb{R}),$$

where $\text{ev} : \mathcal{L} \rightarrow M$ denotes the evaluation of a loop at the point $0 \in S^1 = \mathbb{R}/\mathbb{Z}$.

Definition 2.1. A c-structure for (M, ω) is a connected component of the space of \mathbb{R} -vector bundle homomorphisms $\tilde{y} : S^1 \times \mathbb{C}^n \rightarrow TM$, satisfying $\omega(\tilde{y}_t(v), \tilde{y}_t(iv)) > 0$, for all $t \in S^1$ and all $0 \neq v \in \mathbb{C}^n$.² We denote by $[\tilde{y}]$ the c-structure represented by \tilde{y} .

By assigning to \tilde{y} the underlying map $y : S^1 \rightarrow M$ one obtains a surjective mapping $\pi : \{\text{c-structures}\} \rightarrow \pi_0(\mathcal{L})$, and we say that $[\tilde{y}]$ is a c-structure above $\gamma = \pi([\tilde{y}])$. The group $\pi_1(\text{GL}_n(\mathbb{C})) \cong \mathbb{Z}$ acts freely on the set of c-structures with $\pi_0(\mathcal{L})$ as the set of orbits. By fixing a c-structure $[\tilde{y}]$ we specify a homotopy class of closed curves γ or equivalently a component \mathcal{L}_γ of \mathcal{L} . By choosing a representative \tilde{y} of $[\tilde{y}]$ we specify a base point y in \mathcal{L}_γ .

Note, that if γ is trivial, i.e. \mathcal{L}_γ is the component of contractible loops, then there exists a canonic c-structure above γ , given by any $\tilde{y} : S^1 \times \mathbb{C}^n \rightarrow TM$, which is constant in $t \in S^1$.

We choose a c-structure $[\tilde{y}]$ and set $\gamma = \pi([\tilde{y}])$. Consider the connected Abelian principal covering $\pi : \widetilde{\mathcal{L}}_\gamma \rightarrow \mathcal{L}_\gamma$, with structure group

$$(2.1) \quad \Gamma := \frac{\pi_1(\mathcal{L}, y)}{\ker \bar{c}_1 \cap \ker \phi},$$

where \bar{c}_1 and ϕ are considered as homomorphisms $\pi_1(\mathcal{L}, y) \rightarrow \mathbb{R}$. Clearly $\pi^* \phi = \pi^* \bar{c}_1 = 0 \in H^1(\widetilde{\mathcal{L}}_\gamma; \mathbb{R})$, and \bar{c}_1 and ϕ induce homomorphisms

$$\bar{c}_1 : \Gamma \rightarrow \mathbb{Z} \quad \text{and} \quad \phi : \Gamma \rightarrow \mathbb{R}.$$

We define the minimal Chern number $N \in \mathbb{N}$ by

$$N\mathbb{Z} = \text{img}(H_1(\mathcal{L}_\gamma; \mathbb{Z}) \xrightarrow{\bar{c}_1} \mathbb{Z}) \subseteq \mathbb{Z},$$

with the convention $N = \infty$ if this image is 0. N depends only on (M, ω, γ) . If γ is trivial this is the usual minimal Chern number as discussed in [HS95], but in general it is smaller. Indeed, the diagram

$$\begin{array}{ccc} H_1(\mathcal{L}_\gamma; \mathbb{Z}) & \xrightarrow{\bar{c}_1} & \mathbb{Z} \\ \uparrow & & \uparrow c_1 \\ \pi_1(\mathcal{L}, y) & \longleftarrow & \pi_2(M) \end{array}$$

commutes, where the bottom mapping $\pi_2(M) = \pi_1(\Omega(M, y(0))) \rightarrow \pi_1(\mathcal{L}, y)$ is induced from the mapping $\Omega(M, y(0)) \rightarrow \mathcal{L}$ given by concatenating a loop based at $y(0)$ with y , and $\Omega(M, y(0))$ denotes the space of base pointed loops.

Finally we choose a commutative ring with unit R and let $\Lambda = \mathcal{N}(\Gamma, \phi, R)$ denote the Novikov ring, associated to Γ and the weighting homomorphism $\phi : \Gamma \rightarrow \mathbb{R}$ with values in R . More precisely Λ consists of all functions $\lambda : \Gamma \rightarrow R$, such that

$$\{A \in \Gamma \mid \phi(A) \leq c, \lambda(A) \neq 0\}$$

²Notice that every such homomorphism is a fiber wise isomorphism.

is finite for all $c \in \mathbb{R}$, with multiplication

$$(\lambda * \nu)(A) = \sum_{B \in \Gamma} \lambda(B) \nu(B^{-1}A).$$

Λ is a commutative ring with unit, which depends on $(M, \omega, \beta, \gamma, R)$. We also consider the group

$$\Gamma_0 := \ker(\bar{c}_1 : \Gamma \rightarrow \mathbb{Z})$$

and the Novikov ring $\Lambda_0 = \mathcal{N}(\Gamma_0, \phi, R)$ associated to $\phi : \Gamma_0 \rightarrow \mathbb{R}$. Clearly Λ_0 is a subring of Λ . If R has no zero divisors then Λ and Λ_0 are rings without zero divisors, for Γ and Γ_0 are free Abelian. If R is a principal ideal domain resp. a field, then so is Λ_0 , since $\phi : \Gamma_0 \rightarrow \mathbb{R}$ is injective, cf. [HS95]

3. ANALYTICAL DATA

Recall that \mathcal{L} is a Fréchet manifold with tangent bundle $C^\infty(S^1, TM)$. So a tangent vector at $x \in \mathcal{L}$ is simply a vector field along x . For $\sigma \in \Omega^k(M)$ we define $\tilde{\sigma} \in \Omega^k(\mathcal{L})$ by

$$\tilde{\sigma}(X_x^1, \dots, X_x^k) := \int_{S^1} \sigma(X_x^1(t), \dots, X_x^k(t)) dt,$$

where X_x^i are vector fields along the common loop x . Every $\tilde{\sigma}$ is S^1 -invariant, i.e. $L_\zeta \tilde{\sigma} = 0$, where ζ is the fundamental vector field³ of the natural S^1 -action on \mathcal{L} , and one has $d\tilde{\sigma} = \widetilde{d\sigma}$. Therefore

$$\Omega^*(M) \rightarrow \Omega^{*-1}(\mathcal{L}), \quad \sigma \mapsto \bar{\sigma} := (-1)^{|\sigma|+1} i_\zeta \tilde{\sigma}$$

commutes with the differentials and every $\bar{\sigma}$ is S^1 -invariant, too. Moreover, for $f : N \rightarrow \mathcal{L}$ and $\sigma \in \Omega^*(M)$ one has

$$(3.1) \quad \int_N f^* \bar{\sigma} = \int_{N \times S^1} \hat{f}^* \sigma,$$

where $\hat{f} : N \times S^1 \rightarrow M$, $\hat{f}(z, t) = f(z)(t)$. Because of (3.1) the induced mapping $H^*(M; \mathbb{R}) \rightarrow H^{*-1}(\mathcal{L}; \mathbb{R})$ maps the first Chern class of (M, ω) to the class \bar{c}_1 we have defined in section 2. Moreover $\tilde{\omega}$ is an S^1 -invariant closed weakly non-degenerate 2-form on \mathcal{L} . From (3.1) we also see, that $\bar{\omega}$ is a form representing $[\bar{\omega}] \in H^1(\mathcal{L}; \mathbb{R})$ from section 2.

Let $Z \in C^\infty(S^1, \mathfrak{X}(M))$ be a 1-periodic vector field. It defines a vector field \tilde{Z} on \mathcal{L} , by setting $\tilde{Z}_x(t) := Z_t(x(t))$, for $x \in \mathcal{L}$ and $t \in S^1$. It is straightforward to show

Lemma 3.1. *If Z is symplectic, i.e. $di_{Z_t}\omega = L_{Z_t}\omega = 0$ for all $t \in S^1$, then \tilde{Z} is symplectic, i.e. $di_{\tilde{Z}}\tilde{\omega} = L_{\tilde{Z}}\tilde{\omega} = 0$. If moreover $[i_{Z_t}\omega] = \beta \in H^1(M; \mathbb{R})$ for all $t \in S^1$ then $[i_{\tilde{Z}}\tilde{\omega}] = \text{ev}^*\beta \in H^1(\mathcal{L}; \mathbb{R})$.*

Suppose Z is symplectic and define the action 1-form

$$(3.2) \quad \alpha := i_{\tilde{Z}}\tilde{\omega} + \bar{\omega} = i_{\tilde{Z}-\zeta}\tilde{\omega} \in \Omega^1(\mathcal{L}).$$

By Lemma 3.1 α is closed, i.e. $\tilde{Z} - \zeta$ is a symplectic vector field on \mathcal{L} . The following is an immediate consequence of the weak non-degeneracy of $\tilde{\omega}$.

³Recall that if $\mu : S^1 \times X \rightarrow X$ is a smooth action, then the fundamental vector field $\zeta \in \mathfrak{X}(X)$ is given by $\zeta(x) := \frac{\partial}{\partial t}|_{t=0}\mu(t, x)$.

Lemma 3.2. *The zeros of α are precisely the 1-periodic solutions $x : S^1 \rightarrow M$ of*

$$(3.3) \quad x'(t) = Z_t(x(t)).$$

If $[i_{Z_t}\omega] = \beta \in H^1(M; \mathbb{R})$ for all $t \in S^1$, then the second part of Lemma 3.1 says, that α is a closed one form representing $\phi \in H^1(\mathcal{L}; \mathbb{R})$. From (2.1) we then see, that $\pi^*\alpha = da$ for some $a \in C^\infty(\widetilde{\mathcal{L}}_\gamma, \mathbb{R})$ and

$$a(A\sharp\tilde{x}) = a(\tilde{x}) + \phi(A),$$

for $\tilde{x} \in \widetilde{\mathcal{L}}_\gamma$ and $A \in \Gamma$.⁴

For $\gamma \in \pi_0(\mathcal{L})$ let \mathcal{P}_γ denote the set of 1-periodic trajectories of (3.3) in \mathcal{L}_γ and set $\tilde{\mathcal{P}}_\gamma := \pi^{-1}(\mathcal{P}_\gamma)$, where $\pi : \widetilde{\mathcal{L}}_\gamma \rightarrow \mathcal{L}_\gamma$ is the covering. If all $x \in \mathcal{P}_\gamma$ are non-degenerate, that is the corresponding fixed points $x(0) \in M$ of the time 1 flow Ψ_1^Z to Z are non-degenerate, then \mathcal{P}_γ is finite.

We denote by \mathfrak{X}^β the 1-periodic time dependent symplectic vector fields $Z \in C^\infty(S^1, \mathfrak{X}(M))$, such that $[i_{Z_t}\omega] = \beta \in H^1(M; \mathbb{R})$, for all $t \in S^1$. Moreover we denote by $\mathfrak{X}_{\gamma\text{-reg}}^\beta$ the subset of \mathfrak{X}^β consisting of vector fields whose 1-periodic trajectories in \mathcal{L}_γ are all non-degenerate.

Proposition 3.3. *Let β and γ be as above. Then the set $\mathfrak{X}_{\gamma\text{-reg}}^\beta$ is a generic subset of \mathfrak{X}^β , in the sense of Baire.*⁵

The genericity of $\mathfrak{X}_{\gamma\text{-reg}}^\beta$ is essentially proven in Theorem 3.1 in [HS95] and Theorem 3.1 in [LO95].

Let $[\tilde{y}]$ be a c-structure set $\gamma = \pi([\tilde{y}])$ and suppose $Z \in \mathfrak{X}_{\gamma\text{-reg}}^\beta$. The c-structure $[\tilde{y}]$ gives rise to a well defined, up to homotopy, \mathbb{C} -vector bundle trivialization of x^*TM for every $x \in \mathcal{L}_\gamma$. So one gets a well defined Conley-Zehnder index $\mu^{[\tilde{y}]} : \tilde{\mathcal{P}}_\gamma \rightarrow \mathbb{Z}$, satisfying

$$\mu^{[\tilde{y}]}(A\sharp\tilde{x}) = \mu^{[\tilde{y}]}(\tilde{x}) + 2\bar{c}_1(A),$$

for all $\tilde{x} \in \tilde{\mathcal{P}}_\gamma$ and $A \in \Gamma$, cf. [SZ92].

Remark 3.4. If one changes the c-structure $[\tilde{y}]$ by an element in $\pi_1(\text{GL}_n(\mathbb{C})) \cong \mathbb{Z}$ (cf. section 2) the component \mathcal{L}_γ remains the same and $\mu^{[\tilde{y}]}$ experiences a shift by a constant in $2\mathbb{N}$. In particular for $\tilde{x}_-, \tilde{x}_+ \in \tilde{\mathcal{P}}_\gamma$ the difference $\mu^{[\tilde{y}]}(\tilde{x}_+) - \mu^{[\tilde{y}]}(\tilde{x}_-)$ depends only on γ . We will drop $[\tilde{y}]$ from the notation and simply write μ for the index. We also have induced index maps $\mu : \tilde{\mathcal{P}}_\gamma \rightarrow \mathbb{Z}_{2N}$ and $\mu : \tilde{\mathcal{P}}_\gamma \rightarrow \mathbb{Z}_2$. Obviously the last one does not depend on the c-structure over γ .

Let C_F^* denote the set of all functions $\xi : \tilde{\mathcal{P}}_\gamma \rightarrow \mathbb{R}$, such that

$$\{\tilde{x} \in \tilde{\mathcal{P}}_\gamma \mid a(\tilde{x}) \leq c, \xi(\tilde{x}) \neq 0\}$$

is finite for all $c \in \mathbb{R}$. C_F^* is a free \mathbb{Z}_{2N} -graded Λ -module, via

$$(\lambda * \xi)(\tilde{x}) = \sum_{A \in \Gamma} \lambda(A) \xi(A^{-1}\sharp\tilde{x}).$$

The component C_F^i , $i \in \mathbb{Z}_{2N}$, consists of the functions ξ which vanish on \tilde{x} with $\mu(\tilde{x}) \neq i$. The total rank of C_F^* equals the number of 1-periodic trajectories of Z in \mathcal{L}_γ , i.e. the cardinality of \mathcal{P}_γ . C_F^* depends on $(M, \omega, [\tilde{y}], Z, R)$, but the associated \mathbb{Z}_2 -graded module does only depend on $(M, \omega, \gamma, Z, R)$, for a change of the c-structure over γ shifts the grading of C_F^* by an even integer.

⁴Note that a is unique up to an additive constant.

⁵This is meant in the same sense, as in [HS95].

4. THE FLOER COMPLEX

A smooth almost complex structure J on M is called compatible with ω if

$$g(X, Y) := \omega(X, JY)$$

defines a Riemannian metric on M . We will denote by \mathcal{J} the set of almost complex structures compatible with ω .

For a closed symplectic manifold (M, ω) , a c-structure $[\tilde{g}]$ above γ and a pair $(Z, J) \in \mathfrak{X}_{\gamma\text{-reg}}^\beta \times \mathcal{J}$ one can associate a cochain complex of free $\Lambda = \mathcal{N}(\Gamma, \phi, R)$ modules. For reason of simplicity we will do this only under the additional hypotheses that (M, ω) is weakly monotone and the pair is γ -regular, a concept defined below. This is not an inconvenient restriction for the study of the closed trajectories of Z because in view of Proposition 4.5(2) for any $Z \in \mathfrak{X}_{\gamma\text{-reg}}^\beta$ there exists a γ -regular pair (Z', J) with Z and Z' having the same trajectories.

Choose $J \in \mathcal{J}$ and let \tilde{g} denote the induced weak Riemannian metric on \mathcal{L} , given by

$$\tilde{g}(X_x, Y_x) = \int_{S^1} g(X_x(t), Y_x(t)) dt.$$

For $Z \in \mathfrak{X}^\beta$ we have the action 1-form α (3.2), whose gradient is a well defined vector field on \mathcal{L} equal to $\tilde{J}\tilde{Z}$, where \tilde{J} is the almost complex structure on \mathcal{L} , $(\tilde{J}X_x)(t) := JX_x(t)$. A gradient flow line is a mapping $u : \mathbb{R} \times S^1 \rightarrow M$ satisfying

$$(4.1) \quad D_{Z,J}u := \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} - JZ_t(u(s, t)) = 0.$$

For every curve $u : \mathbb{R} \rightarrow \mathcal{L}$ we have the energy

$$E(u) := \int_{\mathbb{R}} |u'(s)|_{\tilde{g}}^2 ds = \int_{\mathbb{R} \times S^1} \left| \frac{\partial u}{\partial s} \right|_g^2 ds dt.$$

Using the non-degeneracy of the 1-periodic trajectories in \mathcal{P}_γ one shows, cf. [F89] and [SZ92]:

Proposition 4.1. *Let $Z \in \mathfrak{X}_{\gamma\text{-reg}}^\beta$, $J \in \mathcal{J}$ and suppose $u : \mathbb{R} \rightarrow \mathcal{L}_\gamma$ satisfies (4.1). Then the following are equivalent:*

- (1) $u : \mathbb{R} \rightarrow \mathcal{L}_\gamma$ has finite energy.
- (2) There exist $x_-, x_+ \in \mathcal{P}_\gamma$, such that

$$\lim_{s \rightarrow \pm\infty} u(s, t) = x_\pm(t) \quad \text{and} \quad \lim_{s \rightarrow \pm\infty} \frac{\partial}{\partial s} u(s, t) = 0$$

both uniformly in $t \in S^1$ and exponentially in $s \in \mathbb{R}$.

For $\tilde{x}_-, \tilde{x}_+ \in \tilde{\mathcal{P}}_\gamma$ let $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ denote the space of finite energy solutions of (4.1) connecting \tilde{x}_- with \tilde{x}_+ . If we want to emphasize the dependence on (Z, J) we will write $\mathcal{M}(\tilde{x}_-, \tilde{x}_+, Z, J)$. For $u \in \mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ one has

$$(4.2) \quad E(u) = a(\tilde{x}_+) - a(\tilde{x}_-).$$

Definition 4.2. A compatible almost complex structure $J \in \mathcal{J}$ is called regular if for any simple J -holomorphic curve $v : S^2 \rightarrow M$ the linearization of the Cauchy-Riemann operator $\bar{\partial}_J$ at v is surjective, cf. section 2 in [HS95] for definitions.

We will denote by $\mathcal{J}_{\text{reg}} \subseteq \mathcal{J}$ the set of all regular almost complex structures. It is shown in [HS95] Theorem 2.2, that \mathcal{J}_{reg} is a generic subset of \mathcal{J} in the sense of Baire.

Definition 4.3. A symplectic manifold is called weakly monotone [HS95], if for all $\tau \in \pi_2(M)$

$$3 - n \leq c_1(\tau) < 0 \quad \Rightarrow \quad \omega(\tau) \leq 0,$$

where $\dim(M) = 2n$.

The relevance of this concept comes from the fact, that a weakly monotone manifold, when equipped with $J \in \mathcal{J}_{\text{reg}}$, has no J -holomorphic spheres of negative Chern number. Note that every symplectic manifold of dimension smaller or equal to 6 is weakly monotone.

It is shown in [HS95], Proposition 2.4, that for a closed weakly monotone symplectic manifold and $J \in \mathcal{J}_{\text{reg}}$ the subset $M_0(\infty, J)$ resp. $M_1(\infty, J)$, consisting of points of M which lie on a non-constant J -holomorphic curve $v : S^2 \rightarrow M$ with $c_1(v) \leq 0$ resp. $c_1(v) \leq 1$ has codimension greater or equal to 4 resp. 2, so for generic pairs (Z, J) , both the closed trajectories in the homotopy class γ and the image in M of the relevant maps $u \in \mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ have empty intersections with the sets $M_0(\infty, J)$ and $M_1(\infty, J)$. This is a geometrically convenient feature which permits us to define the Floer complex as in [HS95], but in view of the recent work [LT98] and [FO99], not necessary.

Definition 4.4. Let (M, ω) be a closed weakly monotone symplectic manifold and $\gamma \in \pi_0(\mathcal{L})$. A pair $(Z, J) \in \mathfrak{X}_{\gamma\text{-reg}}^\beta \times \mathcal{J}_{\text{reg}}$ is called γ -regular if the following conditions hold:

- (1) For any $\tilde{x}_-, \tilde{x}_+ \in \tilde{\mathcal{P}}_\gamma$ and any $u \in \mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ the linearization of the Cauchy-Riemann operator $D_{Z,J}$ at u is surjective, cf. [HS95] for definitions.
- (2) For any $x \in \mathcal{P}_\gamma$ and any $t \in S^1$ one has $x(t) \notin M_1(\infty, J)$.
- (3) For any $\tilde{x}_-, \tilde{x}_+ \in \tilde{\mathcal{P}}_\gamma$ with $\mu(\tilde{x}_+) - \mu(\tilde{x}_-) \leq 2$, any $u \in \mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ and any $(s, t) \in \mathbb{R} \times S^1$ one has $u(s, t) \notin M_0(\infty, J)$.

As in [F89], see also the proof of Theorem 3.2 in [HS95], [Mc90], [SZ92] and [LO95] one shows

Proposition 4.5. *Let (M, ω) be a closed weakly monotone symplectic manifold.*

- (1) *The set of γ -regular pairs is a generic subset of $\mathfrak{X}^\beta \times \mathcal{J}$, in the sense of Baire.*
- (2) *For $Z \in \mathfrak{X}_{\gamma\text{-reg}}^\beta$ one can find a γ -regular pair (Z', J) , such that the 1-periodic trajectories of Z and Z' in \mathcal{L}_γ are the same, and the vector fields Z and Z' agree on these trajectories up to order 2. Especially the Conley-Zehnder indices agree, too.*
- (3) *For any γ -regular pair (Z, J) and any $\tilde{x}_-, \tilde{x}_+ \in \tilde{\mathcal{P}}_\gamma$ the space $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ is an orientable smooth manifold of dimension $\mu(\tilde{x}_+) - \mu(\tilde{x}_-)$.⁶ It admits*

⁶Although μ depends on the c-structure $[\tilde{y}]$ over γ , this difference depends only on γ , cf. Remark 3.4.

a natural \mathbb{R} -action given by reparameterization, which is free and proper if the index difference is bigger than 0.

Suppose (M, ω) is weakly monotone and suppose (Z, J) is a γ -regular pair. The uniform bound on the energy (4.2) together with the fact, that due to weak monotonicity no bubbling can occur, yield (cf. [F88], [S90], [HS95], [O95] and [LO95]):

Proposition 4.6. *Suppose $\tilde{x}_-, \tilde{x}_+ \in \tilde{\mathcal{P}}_\gamma$ and $\mu(\tilde{x}_+) - \mu(\tilde{x}_-) = 1$. Then the set $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)/\mathbb{R}$ is a finite. Moreover*

$$\{A \in \Gamma \mid \phi(A) \leq c, c_1(A) = 0, \mathcal{M}(\tilde{x}_-, A\sharp\tilde{x}_+) \neq \emptyset\}$$

is finite for all $c \in \mathbb{R}$.

As in [FH93] we choose coherent orientations \mathcal{O} of $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)/\mathbb{R}$, and for $\mu(\tilde{x}_+) - \mu(\tilde{x}_-) = 1$ we set

$$n(\tilde{x}_-, \tilde{x}_+) := \sharp(\mathcal{M}(\tilde{x}_-, \tilde{x}_+)/\mathbb{R}) \in \mathbb{Z},$$

where the points are counted with signs according to their orientation.

If the index difference is different from 1 we set $n(\tilde{x}_-, \tilde{x}_+) = 0$. The weak monotonicity also implies that for $\mu(\tilde{x}_+) - \mu(\tilde{x}_-) = 2$ the one dimensional manifolds $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)/\mathbb{R}$ are compact up to broken trajectories. Together with a gluing argument the coherent orientations yield

$$(4.3) \quad \sum_{\tilde{x} \in \tilde{\mathcal{P}}_\gamma} n(\tilde{x}_-, \tilde{x})n(\tilde{x}, \tilde{x}_+) = 0$$

and

$$(4.4) \quad n(A\sharp\tilde{x}_-, A\sharp\tilde{x}_+) = n(\tilde{x}_-, \tilde{x}_+),$$

for all $\tilde{x}_-, \tilde{x}_+ \in \tilde{\mathcal{P}}_\gamma$ and all $A \in \Gamma$.

From Proposition 4.6 we see, that

$$\partial(\delta_{\tilde{x}}) := \sum_{\tilde{x}_+ \in \tilde{\mathcal{P}}_\gamma} n(\tilde{x}, \tilde{x}_+)\delta_{\tilde{x}_+} \in C_F^*,$$

and (4.4) gives $\partial(\delta_A * \delta_{\tilde{x}}) = \delta_A * (\partial\delta_{\tilde{x}})$. Here $\delta_{\tilde{x}} \in C_F^*$ denotes the element being 1 at \tilde{x} and zero elsewhere. So ∂ extends uniquely to a Λ -linear map

$$\partial : C_F^* \rightarrow C_F^{*+1}$$

of degree 1, and (4.3) immediately gives $\partial^2 = 0$.

To a closed weakly monotone symplectic manifold (M, ω) , a c-structure $[\tilde{y}]$ over γ , a symplectic vector field $Z \in \mathfrak{X}_{\gamma\text{-reg}}^\beta$, a compatible almost complex structure $J \in \mathcal{J}_{\text{reg}}$, such that the pair (Z, J) is γ -regular, coherent orientations \mathcal{O} and a commutative ring of coefficients R we have associated a \mathbb{Z}_{2N} -graded Floer cochain complex C_F^* of free Λ -modules. We denote the corresponding Floer cohomology Λ -module by H_F^* . A priori it depends on $(M, \omega, [\tilde{y}], Z, J, \mathcal{O}, R)$.

Remark 4.7. As already observed, the Floer cochain complex can be defined even when (M, ω) is not weakly monotone and the pair $(Z, J) \in \mathfrak{X}_{\gamma\text{-reg}}^\beta \times \mathcal{J}$ not necessary γ -regular. In this case, if the pair (Z, J) does not satisfies Definition 4.4(1) (which insures that $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ are smooth orientable manifolds), ‘coherent orientations’ on $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ have to be replaced by ‘compatible oriented Kuranishi structures with corners’ on $\tilde{\mathcal{M}}(\tilde{x}_-, \tilde{x}_+)$ in the sense of Fukaya-Ono [FO99]. Here $\tilde{\mathcal{M}}(\tilde{x}_-, \tilde{x}_+)$ denotes the space of stable trajectories which is compact, Hausdorff and has Kuranishi structures, cf. [FO99], section 19.

5. FLOER COHOMOLOGY AND THE PROOF OF THEOREM 1.1

In this section we want to check that the cohomology of the Floer complex depends only on $(M, \omega, \beta, [\tilde{y}], R)$. We will sketch the proof only in the weakly monotone case, but the statement is true in general, cf. Remark 5.11 below.

Suppose we have two pairs $(Z^i, J^i) \in \mathfrak{X}_{\gamma\text{-reg}}^\beta \times \mathcal{J}$. A homotopy between these pairs is a pair (Z, J) , where $Z \in C^\infty(\mathbb{R}, \mathfrak{X}^\beta)$ and $J \in C^\infty(\mathbb{R}, \mathcal{J})$, such that

$$\begin{aligned} (Z_s, J_s) &= (Z^1, J^1) \quad \text{for all } s \leq -1 \text{ and} \\ (Z_s, J_s) &= (Z^2, J^2) \quad \text{for all } s \geq 1. \end{aligned}$$

Note that for any s the almost complex structure J_s is compatible with ω and we have a smooth family of induced Riemannian metrics g_s on M . For a curve $u : \mathbb{R} \rightarrow \mathcal{L}$ we define the energy by

$$E(u) := \int_{\mathbb{R}} |u'(s)|_{g_s}^2 ds = \int_{\mathbb{R} \times S^1} \left| \frac{\partial u}{\partial s} \right|_{g_s}^2 ds dt = \int_{\mathbb{R} \times S^1} \omega \left(\frac{\partial u}{\partial s}, J_s \frac{\partial u}{\partial s} \right) ds dt.$$

Z provides an s -dependent vector field on \mathcal{L} , whose flow lines $u : \mathbb{R} \times S^1 \rightarrow M$ are the solutions of

$$(5.1) \quad D_{Z,J}u := \frac{\partial u}{\partial s} + J_s \frac{\partial u}{\partial t} - J_s Z_{s,t}(u(s, t)) = 0.$$

As in Proposition 4.1 one shows that for solutions $u : \mathbb{R} \rightarrow \mathcal{L}_\gamma$ of (5.1), having finite energy is equivalent to the existence of $x^1 \in \mathcal{P}_\gamma^1$ and $x^2 \in \mathcal{P}_\gamma^2$, such that $u(s, \cdot)$ converges to x^1 resp. x^2 when $s \rightarrow \infty$ resp. $-\infty$, in the same sense as in Proposition 4.1. For $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$ and $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ we let $\mathcal{M}^{12}(\tilde{x}^1, \tilde{x}^2)$ denote the space of finite energy solutions of (5.1) connecting \tilde{x}^1 with \tilde{x}^2 . If we want to emphasize the dependence on (Z, J) we will write $\mathcal{M}^{12}(\tilde{x}^1, \tilde{x}^2, Z, J)$. For $u \in \mathcal{M}^{12}(\tilde{x}^1, \tilde{x}^2)$ one has an energy estimate (cf. the proof of Theorem 4.3 in [LO95])

$$(5.2) \quad E(u) \leq a^2(\tilde{x}^2) - a^1(\tilde{x}^1) + 2 \max_{\substack{s \in [-1, 1], \, t \in S^1 \\ x \in M}} \left| \frac{\partial h_{s,t}}{\partial s}(x) \right|,$$

where $i_{Z_{s,t}}\omega = i_{Z_t^1}\omega + dh_{s,t}$ and $h_{s,t}(y(0)) = 0$.

Definition 5.1. Let (M, ω) be a closed weakly monotone symplectic manifold. A homotopy (Z, J) between two γ -regular pairs (Z^1, J^1) and (Z^2, J^2) is called γ -regular if the following conditions hold:

- (1) For all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$, $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ and $u \in \mathcal{M}^{12}(\tilde{x}^1, \tilde{x}^2)$ the linearization of the Cauchy-Riemann operator $D_{Z,J}$ at u is surjective, cf. [HS95] for definition.
- (2) $J : \mathbb{R} \rightarrow \mathcal{J}$ is regular in the sense, that

$$\{(s, v) \mid s \in \mathbb{R} \text{ and } v : S^2 \rightarrow M \text{ is } J_s\text{-holomorphic curve}\}$$

is a manifold.

- (3) For all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$, $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ with $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1) \leq 1$, all $u \in \mathcal{M}^{12}(\tilde{x}^1, \tilde{x}^2)$ and all $(s, t) \in \mathbb{R} \times S^1$ one has $u(s, t) \notin M_0(\infty, J_s)$.

As in Proposition 4.5 one shows

Proposition 5.2. *Let (M, ω) be a closed weakly monotone symplectic manifold.*

- (1) *The set of γ -regular homotopies between two given γ -regular pairs is generic in the sense of Baire.*
- (2) *For any γ -regular homotopy the space $\mathcal{M}^{12}(\tilde{x}^1, \tilde{x}^2)$ is an orientable smooth manifold of dimension $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1)$, for all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$ and $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$.⁷*

The uniform energy estimate (5.2) and weak monotonicity gives:

Proposition 5.3. *Let (M, ω) be a closed weakly monotone symplectic manifold and (Z, J) a γ -regular homotopy. Then $\mathcal{M}^{12}(\tilde{x}^1, \tilde{x}^2)$ is a finite set, for all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$, $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ with $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1) = 0$. Moreover*

$$\{A \in \Gamma \mid \phi(A) \leq c, \bar{c}_1(A) = 0, \mathcal{M}^{12}(\tilde{x}^1, A\# \tilde{x}^2) \neq \emptyset\}$$

is finite for all $c \in \mathbb{R}$ and all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$, $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ for which $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1) = 0$.

Using coherent orientations one defines

$$n^{12}(\tilde{x}^1, \tilde{x}^2) := \#(\mathcal{M}^{12}(\tilde{x}^1, \tilde{x}^2)) \in \mathbb{Z},$$

where the points are counted with signs according to their orientation, and if $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1) \neq 0$ we set $n^{12}(\tilde{x}^1, \tilde{x}^2) := 0$. Then one has

$$(5.3) \quad n^{12}(A\# \tilde{x}^1, A\# \tilde{x}^2) = n^{12}(\tilde{x}^1, \tilde{x}^2)$$

and a gluing argument shows

$$(5.4) \quad \sum_{\tilde{x} \in \tilde{\mathcal{P}}_\gamma^1} n^1(\tilde{x}^1, \tilde{x}) n^{12}(\tilde{x}, \tilde{x}^2) = \sum_{\tilde{x} \in \tilde{\mathcal{P}}_\gamma^2} n^{12}(\tilde{x}^1, \tilde{x}) n^2(\tilde{x}, \tilde{x}^2),$$

for all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$, $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ and all $A \in \Gamma$. Proposition 5.3 gives

$$h^{12}(\delta_{\tilde{x}^1}) := \sum_{\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2} n^{12}(\tilde{x}^1, \tilde{x}^2) \delta_{\tilde{x}^2} \in C_F^*(Z^2, J^2).$$

From (5.3) we see that $h^{12}(\delta_A * \delta_{\tilde{x}^1}) = \delta_A * h^{12}(\delta_{\tilde{x}^1})$, and so h^{12} extends uniquely to a Λ -linear map

$$h^{12} : C_F^*(Z^1, J^1) \rightarrow C_F^*(Z^2, J^2)$$

of degree 0. Moreover (5.4) immediately gives $\partial^2 \circ h^{12} = h^{12} \circ \partial^1$, so h^{12} is a chain map and induces

$$h^{12} : H_F^*(Z^1, J^1) \rightarrow H_F^*(Z^2, J^2).$$

Suppose one has two γ -regular homotopies (Z_0, J_0) and (Z_1, J_1) connecting the same γ -regular pairs (Z^1, J^1) and (Z^2, J^2) . Then a homotopy of homotopies between these two γ -regular homotopies is a pair (Z, J) , where $Z \in C^\infty([0, 1] \times \mathbb{R}, \mathfrak{X}^\beta)$ and $J \in C^\infty([0, 1] \times \mathbb{R}, \mathcal{J})$, such that

$$\begin{aligned} (Z_{i,s}, J_{i,s}) &= (Z_i, J_i) \quad \text{for all } s \in \mathbb{R} \text{ and } i = 0, 1, \\ (Z_{\lambda,s}, J_{\lambda,s}) &= (Z^1, J^1) \quad \text{for all } \lambda \in [0, 1], s \leq -1 \text{ and} \\ (Z_{\lambda,s}, J_{\lambda,s}) &= (Z^2, J^2) \quad \text{for all } \lambda \in [0, 1], s \geq 1. \end{aligned}$$

For $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$ and $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ define

$$\mathcal{H}(\tilde{x}^1, \tilde{x}^2) := \{(\lambda, u) \mid \lambda \in [0, 1], u \in \mathcal{M}^{12}(\tilde{x}^1, \tilde{x}^2, Z_\lambda, J_\lambda)\}.$$

⁷Notice that this index difference does not depend on the c-structure above γ , cf. Remark 3.4.

Definition 5.4. A homotopy (Z, J) between two γ -regular homotopies connecting the same two γ -regular pairs as above is called γ -regular if the following holds:

- (1) For all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$ and $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ the space $\mathcal{H}(\tilde{x}^1, \tilde{x}^2)$ is a manifold of dimension $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1) + 1$.
- (2) $J : [0, 1] \times \mathbb{R} \rightarrow \mathcal{J}$ is regular in the sense that

$$\{(\lambda, s, v) \mid \lambda \in [0, 1], s \in \mathbb{R} \text{ and } v : S^2 \rightarrow M \text{ is a } J_{\lambda, s} \text{-holomorphic curve}\}$$

is a manifold.

- (3) For all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$, $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ with $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1) + 1 \leq 1$, all $(\lambda, u) \in \mathcal{H}(\tilde{x}^1, \tilde{x}^2)$ and all $(s, t) \in \mathbb{R} \times S^1$ one has $u(s, t) \notin M_0(\infty, J_{\lambda, s, t})$.

Proposition 5.5. *The set of γ -regular homotopies between two γ -regular homotopies connecting the same γ -regular pairs is generic in the sense of Baire.*

Proposition 5.6. *For a γ -regular homotopy of homotopies (Z, J) the manifold $\mathcal{H}(\tilde{x}^1, \tilde{x}^2)$ is a finite set, for all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$ and $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ with $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1) + 1 = 0$. Moreover*

$$\{A \in \Gamma \mid \phi(A) \leq c, \bar{c}_1(A) = 0, \mathcal{H}(\tilde{x}^1, A\sharp\tilde{x}^2) \neq \emptyset\}$$

is finite for all $c \in \mathbb{R}$ and all $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$, $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$ with $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1) + 1 = 0$.

Using coherent orientations one defines

$$m(\tilde{x}^1, \tilde{x}^2) := \sharp(\mathcal{H}(\tilde{x}^1, \tilde{x}^2)) \in \mathbb{Z}$$

for $\mu^2(\tilde{x}^2) - \mu^1(\tilde{x}^1) + 1 = 0$ and sets $m(\tilde{x}^1, \tilde{x}^2) = 0$ otherwise. Then one has

$$(5.5) \quad m(A\sharp\tilde{x}^1, A\sharp\tilde{x}^2) = m(\tilde{x}^1, \tilde{x}^2)$$

and from a gluing argument one gets

$$(5.6) \quad n_1^{12}(\tilde{x}^1, \tilde{x}^2) - n_0^{12}(\tilde{x}^1, \tilde{x}^2) = \sum_{\tilde{x} \in \tilde{\mathcal{P}}_\gamma^2} \left(m(\tilde{x}^1, \tilde{x}) n^2(\tilde{x}, \tilde{x}^2) + n^1(\tilde{x}^1, \tilde{x}) m(\tilde{x}, \tilde{x}^2) \right).$$

Because of Proposition 5.6 and (5.5)

$$H(\delta_{\tilde{x}^1}) := \sum_{\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2} m(\tilde{x}^1, \tilde{x}^2) \delta_{\tilde{x}^2}$$

extends uniquely to a Λ -linear map $H : C_F^*(Z^1, J^1) \rightarrow C_F^{*-1}(Z^2, J^2)$ of degree -1 . Equation (5.6) shows that H is a chain homotopy, i.e. $h_1^{12} - h_0^{12} = \partial^2 H + H \partial^1$. So

$$(5.7) \quad h_1^{12} = h_0^{12} : H_F^*(Z^1, J^1) \rightarrow H_F^*(Z^2, J^2).$$

Using the constant homotopy, which is regular in the sense of Definition 5.1 one immediately gets $h^{11} = \text{id}$, even on chain level for this special homotopy. Concatenating two homotopies and using a gluing argument one shows $h^{23} \circ h^{12} = h^{13}$, again on chain level if the ambiguity in the concatenation is chosen large enough, cf. Lemma 6.4 in [SZ92]. It follows immediately that (5.7) are canonic isomorphisms and one gets, cf. [F89], [SZ92], [HS95] and [LO95]:

Theorem 5.7. *Let (M, ω) be a closed connected weakly monotone symplectic manifold, $\beta \in H^1(M; \mathbb{R})$, $[\tilde{y}]$ a c -structure above γ , and R a commutative ring with unit. Any γ -regular pair $(Z, J) \in \mathfrak{X}_{\gamma\text{-reg}}^\beta \times \mathcal{J}_{\text{reg}}$ and coherent orientations \mathcal{O} provide a \mathbb{Z}_{2N} -graded cochain complex C_F^* of free Λ -modules. Any γ -regular homotopy between such pairs induces a morphism of cochain complexes and an isomorphism in cohomology. Different homotopies induce chain homotopic morphisms and therefore the cohomologies of all these cochain complexes are canonically isomorphic and will be denoted by*

$$H_F^*(M, \omega, \beta, [\tilde{y}], R).$$

Here Λ is the Novikov ring associated to the group Γ and the weighting homomorphism $\phi : \Gamma \rightarrow \mathbb{R}$, and N is the minimal Chern number defined by $\bar{c}_1(H_1(\mathcal{L}_\gamma; \mathbb{Z})) = N\mathbb{Z}$. The associated \mathbb{Z}_2 -graded module depends only on $(M, \omega, \beta, \gamma, R)$.

Let g be any Riemannian metric on the compact manifold M . For $a \in \Omega^*(M)$ define the supremum norm $\|a\|_0 := \sup_{x \in M} |a_x|$ and consider the norm $|\cdot|$ on $H^*(M, \mathbb{R})$ defined by $|\alpha| := \|a\|_0$, where a is the harmonic representative of $\alpha \in H^*(M; \mathbb{R})$. Let $r(g) > 0$ denote the injectivity radius of (M, g) . If (M, ω) is symplectic we set $\varepsilon(g, \omega) := r(g)/\|\sharp_\omega\|_0 > 0$, where $\sharp_\omega : T^*M \rightarrow TM$ denotes the vector bundle isomorphism induced by ω . Suppose $\beta \in H^1(M; \mathbb{R})$ with $|\beta| \leq \varepsilon(g, \omega)$. Consider the time independent symplectic vector field $Z := \sharp_\omega b$, where $b \in \Omega^1(M)$ is the harmonic representative of β . Then $\|Z\|_0 \leq r(g)$, hence any 1-periodic trajectory of Z must be contractible.

Theorem 5.8. *Let (M, ω) be a closed connected weakly monotone symplectic manifold, γ be a homotopy class of closed curves in M and suppose that at least one of the following conditions is satisfied:*

- (1) $\beta(\gamma) \neq 0$ or
- (2) γ non-trivial and $|\beta| \leq \varepsilon(\omega, g)$ for any Riemannian metric g on M .

Then

$$H_F^*(M, \omega, \beta, [\tilde{y}], R) = 0,$$

for all c -structures $[\tilde{y}]$ above γ and all commutative rings with unit R .

Proof. Any of the conditions immediately implies, that there exists a time independent symplectic vector field Z , $[i_Z \omega] = \beta$, which can not have 1-periodic trajectories in the homotopy class of γ . Thus, for any $J \in \mathcal{J}_{\text{reg}}$ the pair (Z, J) is γ -regular in the sense of Definition 4.4, and the Floer complex $C_F^*(M, \omega, [\tilde{y}], Z, J, \mathcal{O}, R)$ vanishes. In view of Theorem 5.7 we get $H_F^*(M, \omega, \beta, [\tilde{y}], R) = 0$. \square

Remark. It is possible to construct β and γ both non-trivial but $\beta(\gamma) = 0$, so that the Floer cohomology is non-trivial.

Corollary 5.9. *Let (M, ω) be a closed connected weakly monotone symplectic manifold, and suppose γ is a homotopy class of closed curves in M . Moreover let Z be a time dependent symplectic vector field, such that $[i_{Z_t} \omega]$ is a fixed class in $\beta \in H^1(M; \mathbb{R})$ and such that all 1-periodic trajectories in \mathcal{P}_γ are non-degenerate. If $H_F^*(M, \omega, \beta, \gamma, \mathbb{R}) = 0$ then*

$$|\mathcal{P}_\gamma^{\text{even}}| = |\mathcal{P}_\gamma^{\text{odd}}|,$$

where $|\mathcal{P}_\gamma^{\text{even}}|$ resp. $|\mathcal{P}_\gamma^{\text{odd}}|$ denotes the number of elements of even resp. odd Conley-Zehnder index in \mathcal{P}_γ .

Proof. By Proposition 4.5(2) we may assume that there exists $J \in \mathcal{J}$, such that (Z, J) is a γ -regular pair. Choosing coherent orientations \mathcal{O} we get a well defined Floer complex $C_F^*(M, \omega, \beta, \gamma, Z, J, \mathcal{O}, \mathbb{R})$, whose cohomology vanishes by Theorem 5.7. Since $C_F^*(M, \omega, \beta, \gamma, Z, J, \mathcal{O}, \mathbb{R})$ is a free acyclic complex its Euler characteristic must vanish, i.e. $|\mathcal{P}_\gamma^{\text{even}}| = |\mathcal{P}_\gamma^{\text{odd}}|$. \square

From Corollary 5.9 we immediately get

Corollary 5.10. *Let (M, ω) be closed connected weakly monotone, and let Z be any 1-periodic time dependent symplectic vector field, such that $[i_{Z_t}\omega]$ is a fixed class $\beta \in H^1(M; \mathbb{R})$. If Z has a non-degenerate 1-periodic trajectory in the homotopy class γ and $H_F^*(M, \omega, \beta, \gamma, \mathbb{R}) = 0$, then it must have another 1-periodic solution in the same homotopy class, which might of course be degenerate.*

The last two corollaries prove Theorem 1.1 stated in the introduction.

Remark. Both corollaries remain true for non-trivial γ without the hypothesis $H_F^*(M, \omega, \beta, \gamma, \mathbb{R}) = 0$. They were also known to K. Ono.

Remark 5.11. Theorem 5.7 and 5.8 remain true for any closed symplectic manifold, and any pair $(Z, J) \in \mathfrak{X}_{\gamma\text{-reg}}^\beta \times \mathcal{J}$ and therefore Corollary 5.9 and 5.10 remain true for any closed symplectic manifold.

6. SYMPLECTIC TORSION

In this section we introduce a new invariant, the symplectic torsion. It is associated with a closed symplectic manifold (M, ω) , a homotopy class of closed curves γ and a pair $(Z, J) \in \mathfrak{X}_{\gamma\text{-reg}}^\beta \times \mathcal{J}$. The invariant takes value in the Abelian group $\text{Wh}(\Lambda)$ defined below.

Since the theory of Floer complex was considered only for weakly monotone manifolds and for γ -regular pairs, these assumptions will be understood here too, but they are not necessary.

We recall some basic properties and definitions of the Milnor torsion. Let A be (for simplicity) a commutative ring with unit. We consider the Abelian groups

$$K_1(A) := \frac{\text{GL}_\infty(A)}{[\text{GL}_\infty(A), \text{GL}_\infty(A)]} \quad \text{and} \quad \bar{K}_1(A) := K_1(A)/(\pm 1),$$

where $(\pm 1) \in K_1(A)$ are the elements represented by $(\pm 1) \in \text{GL}_1(A)$. Recall that one has a homomorphism $\det : \bar{K}_1(A) \rightarrow U(A)/\{\pm 1\}$, where $U(A)$ denotes the units of A , and if A has no zero divisors this is an isomorphism. We will write $\bar{K}_1(A)$ additively.

Let M be a finitely generated free A -module and suppose $b = \{b_1, \dots, b_r\}$ and $c = \{c_1, \dots, c_r\}$ are two bases of M . Then one has $b_i = \sum_{j=1}^r a_{ij}c_j$ for some $a_{ij} \in A$ and we will write $[b/c] \in \bar{K}_1(A)$ for the element represented by the $r \times r$ -matrix (a_{ij}) . Note that this does not depend on the ordering of the bases, and $[b/d] = [b/c] + [c/d]$ for bases b, c and d .

Now let M^* be a free finitely generated \mathbb{Z}_2 -graded A -module. A graded base for M^* is $b^* = (b^0, b^1)$, where b^i is a base of M^i , $i \in \mathbb{Z}_2$. For two graded bases b^* and c^* we define

$$[b^*/c^*] := [b^0/c^0] - [b^1/c^1] \in \bar{K}_1(A).$$

We call two graded bases b^* and c^* equivalent if $[b^*/c^*] = 0$ and will write \mathbf{b}^* for the equivalence class. Note that $[\mathbf{b}^*/\mathbf{c}^*] \in \bar{K}_1(A)$ is well defined, and one has

$$(6.1) \quad [\mathbf{b}^*/\mathbf{d}^*] = [\mathbf{b}^*/\mathbf{c}^*] + [\mathbf{c}^*/\mathbf{d}^*]$$

for three equivalence classes of graded bases \mathbf{b}^* , \mathbf{c}^* and \mathbf{d}^* on M^* . Suppose one has a short exact sequence of free finitely generated \mathbb{Z}_2 -graded A -modules

$$0 \rightarrow M_1^* \rightarrow M_2^* \rightarrow M_3^* \rightarrow 0$$

and suppose \mathbf{b}_1^* resp. \mathbf{b}_3^* are equivalence classes of graded bases on M_1^* resp. M_3^* . Choosing representing graded bases b_1^* , b_3^* and lifting b_3^* to M_2^* one obtains a graded base $b_1^*b_3^*$ of M_2^* and a well defined equivalence class of graded bases $\mathbf{b}_1^*\mathbf{b}_3^*$ on M_2^* . If $\tilde{\mathbf{b}}_1^*$ and $\tilde{\mathbf{b}}_3^*$ are other equivalence classes of graded bases on M_1^* and M_3^* , one has

$$(6.2) \quad [\tilde{\mathbf{b}}_1^*\tilde{\mathbf{b}}_3^*/\mathbf{b}_1^*\mathbf{b}_3^*] = [\tilde{\mathbf{b}}_1^*/\mathbf{b}_1^*] + [\tilde{\mathbf{b}}_3^*/\mathbf{b}_3^*].$$

Next consider a free finitely generated \mathbb{Z}_2 -graded chain complex of A -modules (C^*, ∂^*) . It gives rise to three \mathbb{Z}_2 -graded A -modules, namely the cycles $Z^* := \ker(\partial^*)$, the boundaries $B^* := \text{img}(\partial^{*-1})$ and the homology $H^* = Z^*/B^*$. Suppose H^* and B^* are also free, and C^* and H^* are equipped with equivalence classes of graded bases \mathbf{c}^* and \mathbf{h}^* , respectively. Choose an equivalence class of graded bases \mathbf{b}^* of B^* . The short exact sequences of \mathbb{Z}_2 -graded A -modules

$$0 \rightarrow B^* \rightarrow Z^* \rightarrow H^* \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Z^* \rightarrow C^* \xrightarrow{\partial^*} B^{*+1} \rightarrow 0$$

give rise to an equivalence class of graded bases $\mathbf{b}^*\mathbf{h}^*$ on Z^* and $(\mathbf{b}^*\mathbf{h}^*)\mathbf{b}^{*+1}$ on C^* .⁸ The Milnor torsion is now defined by, cf. [Mi66],

$$\tau(C^*, \partial^*, \mathbf{c}^*, \mathbf{h}^*) := [(\mathbf{b}^*\mathbf{h}^*)\mathbf{b}^{*+1}/\mathbf{c}^*] \in \bar{K}_1(A).$$

A straight forward calculation using (6.1), (6.2) and $[\mathbf{b}^*/\tilde{\mathbf{b}}^*] = -[\mathbf{b}^{*+1}/\tilde{\mathbf{b}}^{*+1}]$ shows that it does not depend on the choice of \mathbf{b}^* , and if $\tilde{\mathbf{c}}^*$ and $\tilde{\mathbf{h}}^*$ are other equivalence classes of graded bases one gets

$$(6.3) \quad \tau(C^*, \partial^*, \tilde{\mathbf{c}}^*, \tilde{\mathbf{h}}^*) - \tau(C^*, \partial^*, \mathbf{c}^*, \mathbf{h}^*) = [\tilde{\mathbf{h}}^*/\mathbf{h}^*] - [\tilde{\mathbf{c}}^*/\mathbf{c}^*].$$

Suppose one has a short exact sequence of free \mathbb{Z}_2 -graded acyclic chain complexes $0 \rightarrow C_1^* \rightarrow C_2^* \rightarrow C_3^* \rightarrow 0$, such that B_1^* and B_3^* are free with equivalence classes of graded bases \mathbf{c}_1^* and \mathbf{c}_3^* . Then $\mathbf{c}_1^*\mathbf{c}_3^*$ is an equivalence class of graded bases of C_2^* and

$$(6.4) \quad \tau(C_2^*, \partial_2^*, \mathbf{c}_1^*\mathbf{c}_3^*, \emptyset) = \tau(C_1^*, \partial_1^*, \mathbf{c}_1^*, \emptyset) + \tau(C_3^*, \partial_3^*, \mathbf{c}_3^*, \emptyset).$$

To see (6.4) recall that one has a short exact sequence $0 \rightarrow B_1^* \rightarrow B_2^* \rightarrow B_3^*$, choose equivalence classes of graded bases \mathbf{b}_1^* and \mathbf{b}_3^* of B_1^* and B_3^* and equip B_2^* with

⁸If B^* is a \mathbb{Z}_2 -graded module B^{*+1} denotes the same module with shifted grading. So $\partial^* : C^* \rightarrow B^{*+1}$ is of degree 0. If b^* is a graded base for B^* , then b^{*+1} will denote the corresponding graded base of B^{*+1} , and similarly for equivalence classes of graded bases.

$\mathbf{b}_1^* \mathbf{b}_3^*$. By definition we have $\tau(C_2^*, \partial_2^*, \mathbf{c}_1^* \mathbf{c}_3^*, \emptyset) = [(\mathbf{b}_1^* \mathbf{b}_3^*)(\mathbf{b}_1^{*+1} \mathbf{b}_3^{*+1}) / \mathbf{c}_1^* \mathbf{c}_3^*]$. It is easy to see, that $[(\mathbf{b}_1^* \mathbf{b}_3^*)(\mathbf{b}_1^{*+1} \mathbf{b}_3^{*+1}) / (\mathbf{b}_1^* \mathbf{b}_1^{*+1})(\mathbf{b}_3^* \mathbf{b}_3^{*+1})] = 0$. Together with (6.1) and (6.2) this immediately gives (6.4).

Given a chain mapping $f^* : (C_1^*, \partial_1^*) \rightarrow (C_2^*, \partial_2^*)$ between two \mathbb{Z}_2 -graded complexes, we consider the ‘mapping cone’ (C_f^*, ∂_f^*) , a \mathbb{Z}_2 -graded chain complex defined by

$$C_f^* := C_2^* \oplus C_1^{*+1}, \quad \partial_f^* := \begin{pmatrix} \partial_2^* & (-1)^{*+1} f^{*+1} \\ 0 & \partial_1^{*+1} \end{pmatrix}.$$

Note, that one has a short exact sequence $0 \rightarrow C_2^* \rightarrow C_f^* \rightarrow C_1^{*+1} \rightarrow 0$ of \mathbb{Z}_2 -graded complexes. We assume that C_1^* , C_2^* , B_1^* and Z_2^* are free and that f^* induces an isomorphism in homology. Then C_f^* is free, acyclic and $Z_f^* = B_f^*$ is free, too. Given equivalence classes of graded bases \mathbf{c}_1^* and \mathbf{c}_2^* of C_1^* and C_2^* one gets an equivalence class of graded bases $\mathbf{c}_2^* \mathbf{c}_1^{*+1}$ on C_f^* and defines the relative torsion by

$$\tau(f^*, \mathbf{c}_1^*, \mathbf{c}_2^*) := \tau(C_f^*, \partial_f^*, \mathbf{c}_2^* \mathbf{c}_1^{*+1}, \emptyset) \in \bar{K}_1(A).$$

From (6.3) one immediately gets

$$(6.5) \quad \tau(f^*, \tilde{\mathbf{c}}_1^*, \tilde{\mathbf{c}}_2^*) - \tau(f^*, \mathbf{c}_1^*, \mathbf{c}_2^*) = [\tilde{\mathbf{c}}_1^* / \mathbf{c}_1^*] - [\tilde{\mathbf{c}}_2^* / \mathbf{c}_2^*].$$

If f and g are chain homotopic, i.e. there exists $H : C_1^* \rightarrow C_2^{*-1}$ satisfying $f^* - g^* = \partial_2^{*-1} H^* + H^{*+1} \partial_1^*$, then one obtains an isomorphism of chain complexes

$$\psi_H^* : C_f^* = C_2^* \oplus C_1^{*+1} \rightarrow C_2^* \oplus C_1^{*+1} = C_g^*, \quad \psi_H^* = \begin{pmatrix} \text{id}_{C_2^*} & (-1)^{*+1} H^{*+1} \\ 0 & \text{id}_{C_1^{*+1}} \end{pmatrix}.$$

Obviously $\psi_H^*(\mathbf{c}_2^* \mathbf{c}_1^{*+1}) = \mathbf{c}_2^* \mathbf{c}_1^{*+1}$ and hence

$$(6.6) \quad \tau(f^*, \mathbf{c}_1^*, \mathbf{c}_2^*) = \tau(g^*, \mathbf{c}_1^*, \mathbf{c}_2^*)$$

for chain homotopic maps f and g .

Suppose we have $f^* : (C_1^*, \partial_1^*) \rightarrow (C_2^*, \partial_2^*)$ and $g^* : (C_2^*, \partial_2^*) \rightarrow (C_3^*, \partial_3^*)$ as above and equivalence classes of graded bases \mathbf{c}_1^* , \mathbf{c}_2^* and \mathbf{c}_3^* . Consider the chain mappings

$$\psi^* := \begin{pmatrix} g & 0 \\ 0 & \text{id}_{C_1^*} \end{pmatrix} : C_f^* \rightarrow C_{gf}^* \quad \text{and} \quad \phi^* := \begin{pmatrix} gf & 0 \\ 0 & f \end{pmatrix} : C_{\text{id}_{C_1^*}}^* \rightarrow C_g^*$$

These give rise to short exact sequences $0 \rightarrow C_{gf}^* \rightarrow C_\psi^* \rightarrow C_f^{*+1} \rightarrow 0$ and $0 \rightarrow C_g^* \rightarrow C_\phi^* \rightarrow C_{\text{id}_{C_1^*}}^{*+1} \rightarrow 0$. Interchanging the second and third factor defines an isomorphism from C_ψ^* to C_ϕ^* which sends $\mathbf{c}_3^* \mathbf{c}_1^{*+1} \mathbf{c}_2^{*+1} \mathbf{c}_1^{*+2}$ to $\mathbf{c}_3^* \mathbf{c}_2^{*+1} \mathbf{c}_1^{*+1} \mathbf{c}_1^{*+2}$. So $\tau(\psi^*, \mathbf{c}_2^* \mathbf{c}_1^{*+1}, \mathbf{c}_3^* \mathbf{c}_1^{*+1}) = \tau(\phi^*, \mathbf{c}_1^* \mathbf{c}_1^{*+1}, \mathbf{c}_3^{*+1} \mathbf{c}_2^*)$. Together with (6.4) this yields

$$(6.7) \quad \tau(g^* \circ f^*, \mathbf{c}_1^*, \mathbf{c}_3^*) = \tau(f^*, \mathbf{c}_1^*, \mathbf{c}_2^*) + \tau(g^*, \mathbf{c}_2^*, \mathbf{c}_3^*),$$

since obviously $\tau(\text{id}_{C_1^*}, \mathbf{c}_1^*, \mathbf{c}_1^*) = 0$.

If $f^* : (C_1^*, \partial_1^*) \rightarrow (C_2^*, \partial_2^*)$ is a chain mapping between acyclic \mathbb{Z}_2 -graded complexes, such that B_1^* , B_2^* are free, then the short exact sequence $0 \rightarrow C_2^* \rightarrow C_f^* \rightarrow C_1^{*+1} \rightarrow 0$ and (6.4) yield

$$(6.8) \quad \tau(f^*, \mathbf{c}_1^*, \mathbf{c}_2^*) = \tau(C_2^*, \partial_2^*, \mathbf{c}_2^*, \emptyset) - \tau(C_1^*, \partial_1^*, \mathbf{c}_1^*, \emptyset).$$

We will apply the previous definitions to the Floer complex defined in section 4. Recall that the Floer complex was a free \mathbb{Z}_2 -graded cochain complex over the Novikov ring Λ associated to $(M, \omega, \gamma, Z, J, \mathcal{O}, R)$. We will assume that R is a principal ideal domain. Then Λ has no zero divisors and the subring Λ_0 is a principal ideal domain too, see section 2. Note that via $\Gamma \subseteq U(\Lambda) = \mathrm{GL}_1(\Lambda)$ resp. $\Gamma_0 \subseteq U(\Lambda_0) = \mathrm{GL}_1(\Lambda_0)$, Γ resp. Γ_0 becomes a subgroup of $K_1(\Lambda)$ resp. $K_1(\Lambda_0)$. We define the Whitehead groups by

$$\mathrm{Wh}(\Lambda) := \frac{K_1(\Lambda)}{\{\pm\Gamma\}} = \frac{\bar{K}_1(\Lambda)}{\Gamma} \quad \text{and} \quad \mathrm{Wh}(\Lambda_0) := \frac{K_1(\Lambda_0)}{\{\pm\Gamma_0\}} = \frac{\bar{K}_1(\Lambda_0)}{\Gamma_0}.$$

Note that since Λ and Λ_0 have no zero divisors we have isomorphisms

$$\det : \mathrm{Wh}(\Lambda) \cong U(\Lambda)/\{\pm\Gamma\} \quad \text{and} \quad \det : \mathrm{Wh}(\Lambda_0) \cong U(\Lambda_0)/\{\pm\Gamma_0\}.$$

So the inclusion $i : \Lambda_0 \rightarrow \Lambda$ induces an injective homomorphism $\mathrm{Wh}(i) : \mathrm{Wh}(\Lambda_0) \rightarrow \mathrm{Wh}(\Lambda)$.

The relative torsion $\tau(\gamma, (Z_1, J_1), (Z_2, J_2))$. To the data $(M, \omega, \gamma, Z, J, \mathcal{O}, R)$, where (Z, J) is γ -regular, we have in (section 4) associated the Floer complex C_F^* , a free \mathbb{Z}_2 -graded cochain complex of Λ -modules. Choosing a lift $\tilde{x} \in \tilde{\mathcal{P}}_\gamma$ for every $x \in \mathcal{P}_\gamma$ defines a base $\{\delta_{\tilde{x}} \mid x \in \mathcal{P}_\gamma\}$ of C_F^* . One can choose them, such that the matrix expression of $\partial^0 : C_F^0 \rightarrow C_F^1$ has entries in Λ_0 . So there exist free Λ_0 -modules \bar{C}_F^0 , \bar{C}_F^1 and $\bar{\partial}^0 : \bar{C}_F^0 \rightarrow \bar{C}_F^1$, such that

$$C_F^0 = \Lambda \otimes_{\Lambda_0} \bar{C}_F^0, \quad C_F^1 = \Lambda \otimes_{\Lambda_0} \bar{C}_F^1 \quad \text{and} \quad \partial^0 = \Lambda \otimes_{\Lambda_0} \bar{\partial}^0.$$

Since Λ_0 is a principal ideal domain we see that $Z_F^0 = \Lambda \otimes_{\Lambda_0} \bar{Z}_F^0$ and $B_F^1 = \Lambda \otimes_{\Lambda_0} \bar{B}_F^1$ are free Λ -modules. Similarly, but one has to choose the lifts differently, one shows that Z_F^1 and B_F^0 are free Λ -modules.

For any two γ -regular pairs (Z_1, J_1) and (Z_2, J_2) with $\beta_1 = \beta_2 \in H^1(M; \mathbb{R})$, we have (in section 5) constructed chain mappings $h^{12} : C_{1,F}^* \rightarrow C_{2,F}^*$ and all of them were chain homotopic, cf. Theorem 5.7. For every $x^1 \in \mathcal{P}_\gamma^1$ and $x^2 \in \mathcal{P}_\gamma^2$ choose lifts $\tilde{x}^1 \in \tilde{\mathcal{P}}_\gamma^1$ and $\tilde{x}^2 \in \tilde{\mathcal{P}}_\gamma^2$. So we get bases $\mathbf{c}_1^* := \{\delta_{\tilde{x}^1} \mid x^1 \in \mathcal{P}_\gamma^1\}$ of $C_F^*(Z_1, J_1)$ and $\mathbf{c}_2^* := \{\delta_{\tilde{x}^2} \mid x^2 \in \mathcal{P}_\gamma^2\}$ of $C_F^*(Z_2, J_2)$. For two γ -regular pairs (Z_1, J_1) and (Z_2, J_2) with $\beta_1 = \beta_2 \in H^1(M; \mathbb{R})$ we define the relative torsion by

$$\tau(\gamma, (Z_1, J_1), (Z_2, J_2)) := \tau(h^{12}, \mathbf{c}_1^*, \mathbf{c}_2^*) \in \mathrm{Wh}(\Lambda).$$

By (6.6) this does not depend on h^{12} and by (6.5) it does not depend on the lifts \tilde{x}^1 and \tilde{x}^2 , for different choices give rise to different bases $\tilde{\mathbf{c}}_1^*$, $\tilde{\mathbf{c}}_2^*$ which obviously satisfy $[\tilde{\mathbf{c}}_1^*, \mathbf{c}_1^*] \in \Gamma$ and $[\tilde{\mathbf{c}}_2^*, \mathbf{c}_2^*] \in \Gamma$. From (6.7) and the remarks right before Theorem 5.7 we get

$$\tau(\gamma, (Z_1, J_1), (Z_3, J_3)) = \tau(\gamma, (Z_1, J_1), (Z_2, J_2)) + \tau(\gamma, (Z_2, J_2), (Z_3, J_3)),$$

for every γ -regular pairs (Z_1, J_1) , (Z_2, J_2) and (Z_3, J_3) , for which $\beta_1 = \beta_2 = \beta_3 \in H^1(M; \mathbb{R})$.

Remark 6.1. Since $\phi : \Gamma_0 \rightarrow \mathbb{R}$ is injective every $\lambda \in \Lambda_0$ has a well defined ‘leading term’ $\lambda(g) \in R$, where $g \in \Gamma_0$ is the unique element in Γ_0 , such that $\lambda(g) \neq 0$ and $\phi(g)$ is minimal with that property. If $\lambda = 0$ we define it to be $0 \in R$. In [HS95] it is shown that $\lambda \in \Lambda_0$ is invertible if and only if this leading term is invertible in R . So one obtains a homomorphism

$$\mathrm{lt} : U(\Lambda_0)/\{\pm\Gamma_0\} \rightarrow U(R)/\{\pm 1\}.$$

A little inspection of the Floer complex and Remark 6.1 above permits to show

Proposition 6.2.

- (1) *The torsion $\tau(\gamma, (Z_1, J_1), (Z_2, J_2))$ lies in the image of the injective homomorphism $\text{Wh}(\Lambda_0) \rightarrow \text{Wh}(\Lambda)$.*
- (2) *The homomorphism $\text{Wh}(\Lambda_0) \xrightarrow{\det} U(\Lambda_0)/\{\pm\Gamma_0\} \xrightarrow{\text{lt}} U(R)/\{\pm 1\}$ sends the torsion $\tau(\gamma, (Z_1, J_1), (Z_2, J_2))$ to 1.*

The symplectic torsion for $H_F^*(M, \omega, \beta, \gamma, R) = 0$. Since the Floer complex is acyclic we can consider the Milnor torsion associated to the Floer complex generated by the γ -regular pair (Z, J) and coherent orientations \mathcal{O} . Together with the choice of the lifts \tilde{x} these data will lead to an element in $\text{Wh}(\Lambda)$ denoted by

$$\tau(\gamma, (Z, J)) \in \text{Wh}(\Lambda),$$

independent of orientations, lifts and the c-structure above γ . The relation to the relative torsion is given by, see (6.8),

$$\tau(\gamma, (Z_1, J_1), (Z_2, J_2)) = \tau(\gamma, (Z_2, J_2)) - \tau(\gamma, (Z_1, J_1)),$$

for all γ -regular pairs with $\beta_1 = \beta_2 \in H^1(M; \mathbb{R})$. If (Z', J') is a γ -regular pair whose vector field Z' has no 1-periodic trajectories in the class γ , then $\tau(\gamma, (Z', J')) = 0$ and

$$\tau(\gamma, (Z, J)) = \tau(\gamma, (Z', J'), (Z, J)),$$

for every γ -regular pair (Z, J) , with $\beta = \beta' \in H^1(M; \mathbb{R})$.

In section 8 below we will show that this symplectic torsion is in general non-trivial. The non-triviality of this torsion assures the existence of closed trajectories of Z in the class γ , and the non-existence of such trajectories implies the vanishing of this torsion.

The symplectic torsion for γ trivial. Denote by γ_0 the trivial class, i.e. the class of contractible loops.

Definition 6.3. Suppose $(Z, J) \in \mathcal{Y}_{\gamma_0\text{-reg}}^\beta \times \mathcal{J}_{\text{reg}}$, set $\alpha := i_Z \omega$ which is a closed one form and let g be the Riemannian metric induced by J and ω , cf. section 4. The pair (Z, J) is called special if the following condition holds:

- (1) The pair (α, g) is Morse-Smale⁹ and the unstable sets of the critical points with respect to the vector field $\text{grad}_g \alpha$ provide a cell structure of M .

For the reader unfamiliar with the Morse-Smale pairs (1) can be replaced by

- (1') There exists a smooth triangulation of M so that the unstable sets associated with the critical points of α with respect to the vector field $\text{grad}_g \alpha$ identify to the open simplexes of the triangulation.

Lemma 6.4. *For all $\beta \in H^1(M; \mathbb{R})$ there exist special pairs.*

As in [FO99] one can define a Floer complex every pair $(Z, J) \in \mathcal{X}_{\gamma_0\text{-reg}}^\beta \times \mathcal{J}_{\text{reg}}$ and Theorem 5.7 remains true, cf. Remark 5.11. So the relative torsion is also well defined for two such pairs and Proposition 6.2 still holds. In section 7 we will prove

⁹The pair (α, g) consisting of a Riemannian metric g and a closed one form is Morse-Smale if all critical points of α are non-degenerate and for any two such critical points x and y the stable manifold of x and the unstable manifold of y intersect transversally, cf. [BH01].

Proposition 6.5. *Given two special pairs (Z_1, J_1) and (Z_2, J_2) the relative torsion $\tau(\gamma_0, (Z_1, J_1), (Z_2, J_2))$ vanishes.*

Now given an arbitrary γ_0 -regular pair (Z, J) , we define

$$\tau(\gamma_0, (Z, J)) := \tau(\gamma, (Z', J'), (Z, J)),$$

where (Z', J') is any special γ_0 -regular pair. Proposition 6.5 shows, that this definition is independent of the special pair (Z', J') . Proposition 6.2 shows, that the torsion $\tau(\gamma_0, (Z, J))$ is actually an element in $\text{Wh}(\Lambda_0)$ with vanishing leading term.

As indicated in introduction, if Z is time independent and γ is trivial, then this torsion can be calculated. It also has interesting geometric interpretations. This will be presented in a future paper.

7. PROOF OF PROPOSITION 6.5

Let us first recall a few elements of Novikov theory, cf. [BH01]. Let M be closed oriented manifold and let α be a Morse form, i.e. α is closed and all critical points (zeros) are non-degenerated. Let $\beta \in H^1(M; \mathbb{R})$ denote the cohomology class represented by α and suppose we have a connected principal Abelian covering $\pi : \tilde{M} \rightarrow M$, such that $\pi^*\beta = 0 \in H^1(\tilde{M}; \mathbb{R})$. Let Δ denote the structure group and note, that β induces a homomorphism $\beta : \Delta \rightarrow \mathbb{R}$. Choose a commutative ring with unit R and let $\mathcal{N} := \mathcal{N}(\Delta, \beta, R)$ denote the corresponding Novikov ring. The underlying \mathcal{N} -module of the Morse-Novikov complex C_N^* is the set of functions from $\tilde{C} := \pi^{-1}(C)$ to R which satisfy a Novikov condition as in section 3. Here C denotes the critical points of α . C_N^* is a free \mathcal{N} -module of rank equal to the cardinality of C and it is \mathbb{Z} -graded by the Morse index.

Now choose a Riemannian metric g on M and assume that $\text{grad}_g \alpha$ is Morse-Smale. Finally choose orientations \mathcal{O} for the unstable manifolds. Counting the trajectories of $\pi^*(\text{grad}_g \alpha)$ connecting points of index difference 1 in \tilde{C} with appropriate signs defines an \mathcal{N} -linear differential ∂ of degree 1 on C_N^* . This complex is called Morse-Novikov complex and we will denote it for simplicity by $C_N^*(\alpha, g)$, although it also depends on the orientations \mathcal{O} , the covering \tilde{M} and the ring R .

Given two Morse-Smale pairs (α_1, g_1) and (α_2, g_2) a homotopy between them is a smooth family of 1-forms α_t which all represent the same cohomology class β and a smooth family of Riemannian metrics g_t , such that $(\alpha_t, g_t) = (\alpha_1, g_1)$ for $t \leq 1$ and $(\alpha_t, g_t) = (\alpha_2, g_2)$ for $t \geq 2$. If the homotopy satisfies a Morse-Smale condition one can now count the number of trajectories of the time dependent vector field $\pi^*(\text{grad}_{g_t} \alpha_t)$ connecting a critical point of α_1 and a critical point of α_2 which have the same index. This defines a chain mapping $h_N : C_N^*(\alpha_1, g_1) \rightarrow C_N^*(\alpha_2, g_2)$. Let $C_{h_N}^*$ denote the mapping cone of h_N , cf. section 6.

Let (K, L) be a pair of finite CW-complexes, let \tilde{K} denote the universal cover of K with the canonic cell structure and let \tilde{L} be the preimage of L in \tilde{K} . Then $\pi_1(K)$ acts freely on (\tilde{K}, \tilde{L}) and the cellular cochain complex (with compact support) $C_c^*(\tilde{K}, \tilde{L})$ ¹⁰ can be viewed as free finitely generated $R[\pi_1(K)]$ -complex, where $R[\pi_1(K)]$ denotes the group ring of $\pi_1(K)$ with values in R .

We call a Morse-Smale pair (α, g) special if the unstable manifolds of $\text{grad}_g \alpha$ provide a cell decomposition of M .

¹⁰The component $C_c^k(\tilde{K}, \tilde{L})$ consists of the functions with finite support defined on the set of k -cells of \tilde{K} which do not belong to \tilde{L} .

Proposition 7.1. *Any Morse-Smale homotopy (α_t, g_t) between two special Morse-Smale pairs (α_1, g_1) and (α_2, g_2) provides a cell decomposition (K, L) of the compact pair $(M \times [-\infty, \infty], M \times \{-\infty\})$. Moreover*

$$C_{h_N}^* = \mathcal{N} \otimes_{R[\pi_1(M)]} C_c^*(\tilde{K}, \tilde{L})$$

as free \mathbb{Z} -graded \mathcal{N} -complexes. Here \mathcal{N} is regarded as $R[\pi_1(M)]$ -module via the homomorphism $R[\pi_1(M)] \rightarrow \mathcal{N}$ induced by the projection $\pi_1(M) \rightarrow \Delta$.

Now let (Z, J) be a special pair in the sense of Definition 6.3. We then have an associated special pair (α, g) , where $\alpha := i_Z \omega$ and g is the Riemannian metric defined by J and ω , cf. section 4. Let $\iota : M \rightarrow \mathcal{L}_o$ denote the map that sends points in M to constant curves. Moreover let $\tilde{M} \rightarrow M$ be a connected component of the pull back covering $\iota^* \tilde{\mathcal{L}}_o$ and let $\tilde{\iota} : \tilde{M} \rightarrow \tilde{\mathcal{L}}_o$ be the restriction of the natural map $\iota^* \tilde{\mathcal{L}}_o \rightarrow \tilde{\mathcal{L}}_o$. Note that $\iota_* : \pi_1(M) \rightarrow \pi_1(\mathcal{L}_o)$ induces a homomorphism $\Delta \rightarrow \Gamma$, where Δ resp. Γ is the structure group of \tilde{M} resp. $\tilde{\mathcal{L}}_o$, and $\tilde{\iota}$ is equivariant.

Since $Z \in \mathcal{Y}_{\gamma_0\text{-reg}}^\beta \subset \mathcal{X}_{\gamma_0\text{-reg}}^\beta$ every 1-periodic trajectory of Z is constant, i.e. we have a bijection $\iota : C \rightarrow \mathcal{P}_{\gamma_0}$. Moreover the Conley-Zehnder index of a constant trajectory equals the Morse index modulo 2 and hence we obtain an isomorphism of \mathbb{Z}_2 -graded Λ -modules

$$(7.1) \quad C_F^*(Z, J) = \Lambda \otimes_{\mathcal{N}} C_N^*(\alpha, g).$$

As in [FO99]¹¹ one shows that this actually is an isomorphism of cochain complexes, although there might be time dependent tubes connecting two points in C , but they do not contribute to the differential of $C_F^*(Z, J)$.

Suppose (Z_1, J_1) and (Z_2, J_2) be two special pairs and let (α_1, g_1) and (α_2, g_2) denote the associated special Morse-Smale pairs. Moreover let (α_t, g_t) be a Morse-Smale homotopy from (α_1, g_1) to (α_2, g_2) and let

$$h_N : C_N^*(\alpha_1, g_1) \rightarrow C_N^*(\alpha_2, g_2) \quad \text{resp.} \quad h_F : C_F^*(Z_1, J_1) \rightarrow C_F^*(Z_2, J_2)$$

denote the chain mappings induced by (α_t, g_t) resp. (α_t, g_t) regarded as homotopy between (Z_1, J_1) to (Z_2, J_2) . As in [FO99] one shows that up to the isomorphism (7.1) one has

$$h_F = \Lambda \otimes_{\mathcal{N}} h_N.$$

Again, there might be time dependent tubes connecting points in C_1 and C_2 , but they do not contribute to h_F . So we have shown

Proposition 7.2. *In the situation above one has for the mapping cone*

$$C_{h_F}^* = \Lambda \otimes_{\mathcal{N}} C_{h_N}^*$$

as free \mathbb{Z}_2 -graded Λ -complexes.

Actual proof of Proposition 6.5. We continue to use the notation introduced above. From Proposition 7.1 and 7.2 we obtain

$$(7.2) \quad C_{h_F}^* = \Lambda \otimes_{R[\pi_1(M)]} C_c^*(\tilde{K}, \tilde{L})$$

¹¹At this point one requires the work of Fukaya-Ono even in the case of weakly monotone manifolds.

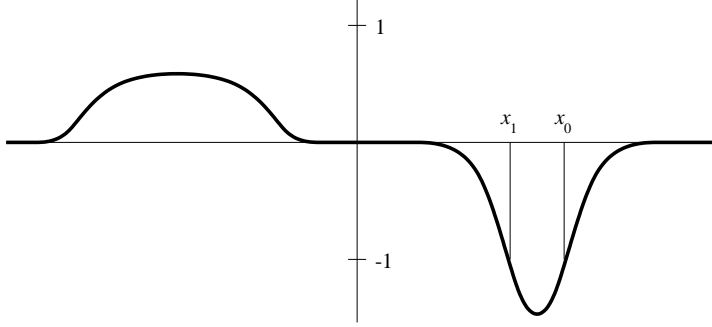
as free \mathbb{Z}_2 -graded Λ -complexes, where (K, L) is a CW-complex with underlying space $(M \times [-\infty, \infty], M \times \{-\infty\})$ and the $R[\pi_1(M)]$ -module structure on Λ is given by the composition $\psi : R[\pi_1(M)] \rightarrow \mathcal{N} \rightarrow \Lambda$. Since L is a deformation retract of K one has a Whitehead torsion $\tau(K, L) \in \text{Wh}(\pi_1(M))$, cf. [Mi66]. From (7.2) one obtains

$$(7.3) \quad \tau(\gamma_0, (Z_1, J_1), (Z_2, J_2)) = \text{Wh}(\psi)(\tau(K, L)) \in \text{Wh}(\Lambda),$$

where $\text{Wh}(\psi) : \text{Wh}(\pi_1(M)) \rightarrow \text{Wh}(\Lambda)$ is the homomorphism induced from ψ . The topological invariance for Whitehead torsion in the case compact manifolds states that $\tau(K, L)$ does only depend on the total space $(M \times [-\infty, \infty], M \times \{-\infty\})$ and not on the cell structure. Using a cylindrical cell structure it follows easily, that $\tau(K, L) = 0 \in \text{Wh}(\pi_1(M))$. So by (7.3) the relative torsion $\tau(\gamma_0, (Z_1, J_1), (Z_2, J_2))$ vanishes too. \square

8. APPENDIX (AN EXAMPLE)

Consider the torus $T = S^1 \times S^1$ with the standard metric $dx^2 + dy^2$, the standard symplectic form $dx \wedge dy$ and the standard complex structure J . Choose bump functions λ and ν , such that $\nu(0) = 1$, $\nu'(0) = 0$, $\nu''(0) = -1$ and such that λ' looks like



where $\lambda'(x_0) = \lambda'(x_1) = -1$. We also assume, that $0 < |\lambda''(x_i)| < 2\pi$ and $0 < |\lambda(x_i)| < 2\pi$. Consider the time dependent Hamiltonian $h_t(x, y) := \lambda(x)\nu(y-t)$ and the corresponding time dependent Hamiltonian vector field

$$Z_t = \lambda(x)\nu'(y-t)\partial_x - \lambda'(x)\nu(y-t)\partial_y.$$

One easily sees, that the closed curves $t \mapsto (x_i, t)$, $i = 0, 1$ are non-degenerate 1-periodic solutions of index difference 1. Indeed, the index in this example is the Maslov index of the path

$$t \mapsto e^{it} \begin{pmatrix} 0 & -\lambda(x_i) \\ -\lambda''(x_i) & 0 \end{pmatrix}$$

which is the Morse index of $J \begin{pmatrix} 0 & -\lambda(x_i) \\ -\lambda''(x_i) & 0 \end{pmatrix}$, since we assumed that $|\lambda''(x_i)| < 2\pi$, $|\lambda(x_i)| < 2\pi$, and this Morse index is 1 or 2, depending on the sign of $\lambda''(x_i)$. These should be all periodic solutions in this homotopy class. However there are lots of degenerate contractible solutions. Moreover we have two connecting orbits $u(s, t) = (x(s), t)$, where x is one of the two (up to shift in s) non-constant solutions of $x'(s) = 1 + \lambda'(x(s))$. There should not exist other connecting orbits. Let $U := \ker \phi$, $R = \mathbb{R}$. Then $\Lambda_0 = \Lambda$ is the field of Laurent series in one variable, say z , and the corresponding \mathbb{Z} -graded Floer complex is:

$$\cdots \rightarrow 0 \rightarrow \Lambda \xrightarrow{1 \pm z} \Lambda \rightarrow 0 \rightarrow \cdots$$

So its torsion is the non-trivial element $1 \pm z \in \frac{\Lambda \setminus \{0\}}{\pm z}$.

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